ON THE STEADY-STATE PROBLEM OF THERMAL CONDUCTION IN A SOLID CYLINDER

(OB ODNOI STATSIONARNOI ZADACHE TEPLOPROVODNOSTI DLIA SPLOSHNOGO TSILINDRA)

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The paper describes a method of constructing the temperature-distribution function for an infinite solid cylinder on whose external surface there exists a temperature distribution $T = f(z)$ extending over half of its length, the other half transferring heat to the surroundings in accordance with Newton's law of cooling.

The solution of the preceding problem reduces itself to the determination of the function $T(r, z)$ which satisfies the Laplace equation in cylindrical coordinates

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{1}$$

with the boundary conditions

$$\frac{\partial T}{\partial r} + hT = 0 \quad \text{for} \quad r = R, \ 0 < z < +\infty \tag{2}$$

$$T = f(z) \quad \text{for} \quad r = R, \ -\infty < z < 0 \tag{3}$$

We shall assume that in the interval $(-\infty, 0)$ the function $f(z)$ can be represented by a Fourier integral, that is by

$$f(z) = \int_{-\infty}^{0} A(\beta) \cos \beta z d\beta, \qquad A(\beta) = \frac{2}{\pi} \int_{-\infty}^{0} f(v) \cos \beta v dv \tag{4}$$

Provisionally, we shall establish that solution of Equation (1) which satisfies the following boundary conditions:

$$\frac{\partial T}{\partial r} + hT = 0 \quad \text{for} \quad r = R, \ 0 < z < +\infty \tag{5}$$

$$T = A \cos \beta z = \frac{1}{2} A (e^{i\beta z} + e^{-i\beta z}) \quad \text{for} \quad r = R, \ -\infty < z < 0 \tag{6}$$

Here, the symbols A and β denote material parameters.

Following [1, 2], we shall introduce the auxiliary solution of Equation (1) in the form

$$T_0(r, z) = BJ_0(mr)\, e^{mz}$$

where m denotes a complex parameter.

Treating B as a function of the parameter $u = mR$, we construct the integral

$$T_1 = \frac{1}{R} \int_C B(u)\, J_0(\rho u)\, e^{\lambda u}\, du \tag{7}$$

for which the contour of integration C is taken along the imaginary axis with circles around the points $\pm i\beta R$, where $z = \lambda R$ and $r = \rho R$.

The integral (7) satisfies Equation (1) if it can be shown that it, together with its derivatives with respect to ρ and λ up to and including second order, converges absolutely and uniformly in the interval

$$\rho < 1 \quad |\lambda| < \infty$$

The solution (7) assumes the following values on the boundary:

$$\frac{\partial T_1}{\partial r} + hT_1 = \frac{1}{R} \int_C K(u)\, e^{\lambda u}\, du \quad \text{for } \rho = 1,\ |\lambda| < \infty \tag{8}$$

$$T_1 = \frac{1}{R} \int_C \psi(u)\, e^{\lambda u}\, du \quad \text{for } \rho = 1,\ |\lambda| < \infty \tag{9}$$

Here

$$B(u) = \frac{RK(u)}{hRJ_0(u) - uJ_1(u)}, \qquad \psi(u) = \frac{RJ_0(u)\, K(u)}{hRJ_0(u) - uJ_1(u)}$$

The function $K(u)$ is so determined as to turn the conditions (8) and (9) into (2) and (3), respectively, namely

$$\int_C K(u)\, e^{\lambda u}\, du = 0 \quad \text{for } \rho = 1,\ \lambda > 0 \tag{10}$$

$$\int_C \psi(u)\, e^{\lambda u}\, du = \frac{1}{2} A\, (e^{i\beta R\lambda} + e^{-i\beta R\lambda}) \quad \text{for } \rho = 1,\ \lambda < 0 \tag{11}$$

The boundary condition (10) will be satisfied if $K(u)$ can be shown to be regular in the domain $\mathrm{Re}(u) < 0$ and to satisfy the conditions of Jordan's lemma in that domain.

In order to satisfy the boundary conditions (11), it is necessary to show that the function $\psi(u)$ is regular in the domain $\operatorname{Re}(u) > 0$, that it satisfies the conditions of Jordan's lemma in that domain, as well as the condition

$$\operatorname{res} \psi(u)e^{\lambda u}|_{u=i\beta R} + \operatorname{res} \psi(u)e^{\lambda u}|_{u=-i\beta R} = -\frac{A}{4\pi i}\left(e^{i\beta R\lambda} + e^{-i\beta R\lambda}\right)$$

Consequently

$$\operatorname{res} K(u)e^{\lambda u}|_{u=i\beta R} + \operatorname{res} K(u)e^{\lambda u}|_{u=-i\beta R} = -\frac{A\left[hI_0(\beta R) + \beta I_1(\beta R)\right]}{4\pi i I_0(\beta R)}\left(e^{i\beta R\lambda} + e^{-i\beta R\lambda}\right)$$

It was shown in [1, 2] that in order to construct the function $K(u)$ it is possible to consider the infinite product

$$\Pi(u) = \prod_{k=1}^{\infty} \frac{1 - u/a_k}{1 - u/b_k}$$

Here a_k and b_k denote the positive roots of the equations

$$hRJ_0(u) - uJ_1(u) = 0, \qquad J_0(u) = 0 \tag{12}$$

Correspondingly, according to [2], we have

$$\Pi(u) \approx \sqrt{-u/hR} \tag{13}$$

for sufficiently large values of $|u|$ in the interval $0 < \delta \leqslant \arg u \leqslant 2\pi - \delta$.

First, it is not difficult to establish that

$$K(u) = -\frac{A\Pi(u)\left[hI_0(\beta R) + \beta I_1(\beta R)\right]}{4\pi i I_0(\beta R)}\left[\frac{1}{(u - i\beta R)\Pi(i\beta R)} + \frac{1}{(u + i\beta R)\Pi(-i\beta R)}\right]$$

satisfies the conditions enumerated previously, and that the function

$$T_1 = \int_C \frac{J_0(\rho u)K(u)e^{\lambda u}}{hRJ_0(u) - uJ_1(u)}\,du = A\varphi(\rho, \lambda, \beta) \tag{14}$$

Here, the function

$$\varphi(\rho, \lambda, \beta) = -\frac{1}{2\pi}\int_{-\infty}^{0}\left[\frac{I_0(\rho z)}{I_0(z)}\operatorname{Im}\left\{\frac{1}{\Pi(-iz)}\left[\frac{\Pi(-i\beta R)}{z - \beta R} + \frac{\Pi(i\beta R)}{z + \beta R}\right]e^{i\lambda z}\right\} - \right.$$
$$\left. - \frac{I_0(\rho z)}{I_0(z)}\left(\frac{1}{z - \beta R} + \frac{1}{z + \beta R}\right)\sin\lambda z\right]dz + \frac{I_0(\rho z)}{I_0(z)}\cos\lambda\beta R$$

is a solution of Equation (1) and satisfies the boundary conditions (5) and (6).

Regarding A as a function of the parameter β, we construct the integral

$$T(\rho, \lambda) = \int_{-\infty}^{0} A(\beta)\, \varphi(\rho, \lambda, \beta)\, d\beta \tag{15}$$

where $A(\beta)$ is determined by Equation (4).

The function (15) satisfies Equation (1) and the boundary conditions (2) and (3).

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